Ergodic Theory and Measured Group Theory
Lecture 25

Examples (continued).

(d) The translation action $\mathbb{Q} \curvearrowright \mathbb{R}$ is ergodic w.r.t. the Lebesgue measure, hence the orbit eq. rel., known as the Vitali eq. rel., denoted E_V, is nonsmooth.

Note that smoothness is closed downward under Borel reductions, i.e. if F is a smooth eq. rel. and $E \leq_B F$, then E too is smooth. Thus, if, say, $\mathbb{E}_0 \leq_B E$ then E is nonsmooth.

$\mathbb{E}_0$-dichotomy (Harrington-Kechris-Louveau). For any Borel eq. rel E.

either E is smooth

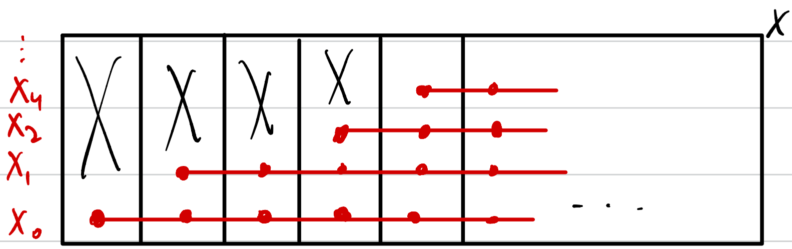
or $\mathbb{E}_0 \sqsubseteq_B E$.

This generalized earlier theorems (in functional analysis) of Glimm and Effros, so people still refer to this as the Glimm–Effros dichotomy.

Corollary. If a Borel eq. rel. E has a measurable reduction to the identity, then it has a Borel reduction (i.e. is smooth).

Proof. We prove the contrapositive: if E is nonsmooth, then $E_0 \sqsubseteq_B E$ hence E cannot have a measurable reduction to the identity because E_0 doesn't have it. $\square$

N-characterization of smoothness. A CBER E on a st. Borel X is smooth $\iff$ $\exists$ Borel $X = \bigsqcup_{n=0}^{\infty} X_n$ s.t. each X_n meets every E-class at most once (i.e. is a partial transversal of E) and for each E-class C, $\{n \in \mathbb{N} : C \cap X_n \neq \emptyset\}$ is an initial segment of $\mathbb{N}$, i.e. an interval $[0,n] \cap \mathbb{N}$ or $\mathbb{N}$.



Proof. $\Rightarrow$. Fix a Borel transversal X_0 for E. Fix Borel involutions (γ_n) of X s.t. $E = \bigcup_n \mathrm{graph}(\gamma_n)$. Suppose $X_0, \dots, X_n$ are defined, and define

$$X_{n+1} := \{\gamma_k \cdot x : x \in X_0 \text{ and } k \text{ is the least s.t. } \gamma_k \cdot x \notin X_0 \cup X_1 \cup \dots \cup X_n\}.$$ $\square$

Corollary. A pmp CBER E is smooth a.e. $\Leftrightarrow$ E is finite a.e. (i.e. each E-class is finite after throwing out a null set).

Proof. $\Leftarrow$. This holds without pmp and we did this last time.

$\Rightarrow$. Let $Y := \{x \in X : [x]_E \text{ is infinite}\}$. We call Y the **aperiodic part of E**. Let $Y = \bigsqcup_{n=0}^{\infty} X_n$ be as in the previous characterization. Then each X_n is a transversal and $\exists$ Borel bijection $\varphi_n : X_0 \to X_n$ with $\varphi_n x \, E \, x$ $\forall x \in X_0$. Thus, X_n has equal measure to X_0 and since $\exists$ infinitely-many disjoint X_n, X_0 must be null. This implies that Y is null. (Think of $\mathbb{Z} \curvearrowright \mathbb{R}$ by translation and $X_0 := [0,1)$.) $\square$

Let's learn how to compute cost of graphings easier.

Lemma. Let $\vec{G}$ be a locally ctbl Borel directed graph without loops on a st. prob. space (X,μ) and suppose that $\vec{G}$ is pmp (i.e. the connectedness relation, ignoring directions, is pmp).

$$\int_X \mathrm{outdeg}_{\vec{G}}(x)\, d\mu(x) = \int_X \mathrm{indeg}_{\vec{G}}(x)\, d\mu(x).$$

Note. This is true for finite graphs, replacing $\int_X$ with $\sum_X$.

Proof. Let (γ_n) be Borel involutions s.t. $E = \bigcup_n \operatorname{graph}(\gamma_n)$ and $\operatorname{graph}(\gamma_n) \cap \operatorname{graph}(\gamma_m) \subseteq \mathrm{Id}_X$. Then $\operatorname{outdeg}_{\vec{G}}(x) = \sum_{n\in\mathbb{N}} \mathbb{1}_{\vec{G}}(x, \gamma_n x)$, hence

$$\int \operatorname{outdeg}_{\vec{G}}(x)\, d\mu(x) = \int \sum_{n\in\mathbb{N}} \mathbb{1}_{\vec{G}}(x, \gamma_n x)\, d\mu(x)$$

$$[\text{Fubini}] \qquad = \sum_{n\in\mathbb{N}} \int \mathbb{1}_{\vec{G}}(x, \gamma_n x)\, d\mu(x)$$

$$\left[\text{change of variable } x \mapsto \gamma_n x \text{ because of pmp}\right] \qquad = \sum_{n\in\mathbb{N}} \int \mathbb{1}_{\vec{G}}(\gamma_n \cdot x, \gamma_n \cdot \gamma_n \cdot x)\, d\mu(x)$$

$$[\text{Fubini}] \qquad = \int \sum_{n\in\mathbb{N}} \mathbb{1}_{\vec{G}}(\gamma_n x, x)\, d\mu(x)$$

$$= \int \operatorname{indeg}_{\vec{G}}(x)\, d\mu(x). \qquad \square$$

Prop. For any locally ctbl pmp Borel graph G

$$\operatorname{Cost}_\mu(G) = \int_X \operatorname{indeg}_{\vec{G}}(x)\, d\mu(x) = \int_X \operatorname{outdeg}_{\vec{G}}(x)\, d\mu(x),$$

$$\operatorname{Cost}_\mu(G) = \frac{1}{2}\int_X \deg_G(x)\, d\mu(x)$$

for any Borel directing $\vec{G}$ of G.

A **directing** of G is any graph $\vec{G}$ s.t. $\forall (x,y) \in G$, exactly one of (x,y), (y,x) is in $\vec{G}$.

E.g. suppose $X = \mathbb{R}$ with its natural linear order, and for each pair x, y, pick the increasing edge (z_0, z_1) $z_0 < z_1$, $\{z_0, z_1\} = \{x, y\}$.

Proof.

$$\frac{1}{2}\int_X \deg_G(x)\, d\mu(x) = \frac{1}{2}\int \left(\mathrm{indeg}_{\vec{G}}(x) + \mathrm{outdeg}_{\vec{G}}(x)\right) d\mu(x)$$

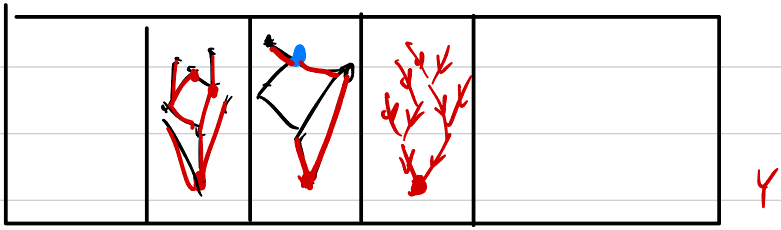
$$= \frac{1}{2} \cdot \left(\int \mathrm{indeg}_{\vec{G}}(x)\, d\mu(x) + \int \mathrm{outdeg}_{\vec{G}}(x)\, d\mu(x) \right)$$

$$= \frac{1}{\cancel{2}} \cdot \cancel{2} \int \mathrm{indeg}_{\vec{G}}(x)\, d\mu(x). \qquad \square$$

Theorem (Levitt). If E is smooth and pmp, then any treeing of E achieves the cost, which is equal to $1 - \mu(Y)$, where Y is any Borel transversal.

In particular, if each E-class has n elements, then $\mathrm{cost}_\mu(E) = 1 - \frac{1}{n}$.

Proof. Let G be any graphing of E and let Y be any transversal. Using a Feldman–Moore edge coloring,



we can talk about lexicographically least shortest paths between point in a Borel fashion. For each $x \in X$, pick the first edge in the lex-least path from x to the unique $y \in Y \cap [x]_E$.

This gives a treeing $H \subseteq G$ of E. Hence $\mathrm{Cost}_\mu(H) \le \mathrm{Cost}_\mu(G)$. Thus it is enough to prove that $\mathrm{Cost}(\text{any treeing}) = 1 - \mu(Y)$.

Let $\vec{H}$ be a directing of H by directing the edges toward Y. This make $\mathrm{outdeg}(x) = 1$ for each $x \in X \setminus Y$, and $\mathrm{outdeg}(y) = 0$ $\forall y \in Y$. Then

$$\mathrm{Cost}_\mu(H) = \int_X \mathrm{outdeg}_{\vec{H}}(x)\, d\mu(x)$$

$$= \int_{X \setminus Y} 1 \, d\mu(x) = \mu(X \setminus Y) = 1 - \mu(Y).$$

$\square$